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Particular solution of DGLAP evolution equation in next-to-leading order and structure functions at low-x

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Abstract : We present particular solutions of singlet and non-singlet Dokshitzer-Gribov-Lipatov Altarelli Parisi (DGLAP) evolution equations in next-to-leading order (NLO) at low-x. We obtain t-evolutions of deuteron, proton, neutron and difference and ratio of proton and neutron structure functions at low-x from DGLAP evolution equations. The results of t-evolutions are compared with HERA and NMC low-x and low-Q^2 data and with those of leading order (LO) solutions of DGLAP evolution equations. We also compare our result of t-evolution of proton structure function with a recent global parameterization

Keywords Particular solution, complete solution, Altarelli-Parisi equation, structure function, low-x physics

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1. Introduction

In a recent paper [1], particular solution of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [2-5] for t and x-evolutions of singlet and non-singlet structure functions in leading order at low-x have been reported. The same technique can be applied to the DGLAP evolution equations in next-to-leading order (NLO) for singlet and non-singlet structure functions to obtain t-evolutions of deuteron, proton, neutron, difference and ratio of proton and neutron structure functions. These NLO results are compared with the HERA H1 [6] and NMC [7] low-x, low-Q^2 data and with those of particular solution in LO and we also compare our results of t-evolution of proton structure functions with recent global parameterization [8].

2. Theory

Though the necessary theory has been discussed elsewhere [9], here we mention some essential steps for clarity. The DGLAP evolution equations with splitting functions [10, 11] for singlet and non-singlet structure functions in NLO are in the standard forms [12]

$$\frac{\partial F_2^S(x,t)}{\partial t}-\frac{\alpha_s(t)}{2\pi}\left[\frac{2}{3}\{3+4\ln(1-x)\}F_2^S(x,t)+\frac{4}{3}\int_x^1\frac{dw}{1-w}\right.$$
$$\times\left\{(1+w^2)F_2^S\left(\frac{x}{w},t\right)-2F_2^S(x,t)\right\}+N_f\int_x^1\left\{w^2+(1-w)^2\right\}$$
$$\left.\times G\left(\frac{x}{w},t\right)dw\right]-\left(\frac{\alpha_s(t)}{2\pi}\right)^2\left[(x-1)F_2^S(x,t)\int_0^1 f(w)dw\right.$$
$$+\int_x^1 f(w)\,F_2^S\left(\frac{x}{w},t\right)dw+\int_x^1 F_{qq}^S(w)\,F_2^S\left(\frac{x}{w},t\right)dw$$
$$\left.+\int_x^1 F_{qg}^S(w)\,G\left(\frac{x}{w},t\right)dw\right]=0 \qquad (1)$$

and

$$\frac{\partial F_2^{NS}(x,t)}{\partial t}-\frac{\alpha_s(t)}{2\pi}\left[\frac{2}{3}\{3+4\ln(1-x)\}F_2^{NS}(x,t)+\frac{4}{3}\int_x^1\frac{dw}{1-w}\right.$$
$$\left.\times\left\{(1+w^2)F_2^{NS}\left(\frac{x}{w},t\right)-2F_2^{NS}(x,t)\right\}\right]-\left(\frac{\alpha_s(t)}{2\pi}\right)^2$$
$$\times\left[(x-1)F_2^{NS}(x,t)\int_0^1 f(w)dw+\int_x^1 f(w)F_2^{NS}\left(\frac{x}{w},t\right)dw\right]=0, \qquad (2)$$

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where

$$\alpha_s(t) = \frac{4\pi}{\beta_0 t}\left[1-\frac{\beta_1 \ln t}{\beta_0^2 t}\right],\ \beta_0 = \frac{33-2n_f}{3}$$

and $\beta_1 = 306-38n_f$

N_f being the number of flavours.

Here, $f(w) = C_F^2\left[P_F(w)-P_A(w)\right]+\frac{1}{2}C_F C_A\left[P_G(w)+P_A(w)\right]$

$+C_F T_R N_f P_{N_f}(w)$

and

$$F_{qq}^S(w) = 2C_F T_R N_f F_{qq}(w)$$

and $\quad F_{qg}^S(w) = C_F T_R N_f F_{qg}^1(w) + C_G T_R N_f F_{qg}^2(w)$.

The explicit forms of higher order kernels are [10, 11]

$$P_F(w) = -\frac{2(1+w^2)}{1-w}\ln w\ln(1-w)-\left(\frac{3}{1-w}+2w\right)\ln w$$

$$-\frac{1}{2}(1+w)\ln^2 w-5(1-w),$$

$$P_G(w) = \frac{1+w^2}{1-w}\left(\ln^2 w+\frac{11}{3}\ln w+\frac{67}{9}-\frac{\pi^2}{3}\right)$$

$$+2(1+w)\ln w+\frac{40}{3}(1-w),$$

$$P_{N_f}(w) = \frac{2}{3}\left[\frac{1+w^2}{1-w}\left(-\ln w-\frac{5}{3}\right)-2(1-w)\right],$$

$$P_A(w) = \frac{2(1+w^2)}{1+w}\int_{w/(1+w)}^{1/(1+w)}\frac{dk}{k}\ln\frac{1-k}{k}+2(1+w)\ln w+4(1-w),$$

$$F_{qq}(w) = \frac{20}{9w}-2+6w-\frac{56}{9}w^2+\left(1+5w+\frac{8}{3}w^2\right)\ln w$$

$$-(1+w)\ln^2 w,$$

$$F_{qg}^1(w) = 4-9w-(1-4w)\ln w-(1-2w)\ln^2 w+4\ln(1-w)$$

$$+\left[2\ln^2\left(\frac{1-w}{w}\right)-4\ln\left(\frac{1-w}{w}\right)-\frac{2}{3}\pi^2+10\right]P_{qg}(w)$$

and

$$F_{qg}^2(w) = \frac{182}{9}+\frac{14}{9}w+\frac{40}{9w}+\left(\frac{136}{3}w-\frac{38}{3}\right)\ln w-4\ln(1-w)$$

$$-(2+8w)\ln^2 w+\left[-\ln^2 w+\frac{44}{3}\ln w-2\ln^2(1-w)+4\ln(1-w)\right.$$

$$\left.+\frac{\pi^2}{3}-\frac{218}{3}\right]P_{qg}(w)+2P_{qg}(-w)\int_{w/1+w}^{1/1+w}\frac{dz}{z}\ln\frac{1-z}{z},$$

where $P_{qg}(w) = w^2+(1-w)^2$, $C_A = C_G = N_C = 3$,

$C_F = \left(N_C^2-1\right)/2N_C$ and $T_R = 1/2$.

Let us introduce the variable $u = 1-w$ and note that [13]

$$\frac{x}{w} = \frac{x}{1-u} = x\sum_{k=0} u^k.$$

The above series is convergent for $|u|<1$. Since $x<w<1$, so $0<u<1-x$ and hence the convergence criterion is satisfied. Now, using Taylor expansion method we can rewrite $F_2^S(x/w,t)$ as

$$F_2^S(x/w,t) = F_2^S\left(x+x\sum_{k=1}^{\infty}u^k, t\right)$$

$$= F_2^S(x,t)+x\sum_{k=1}^{\infty}u^k\frac{\partial F_2^S(x,t)}{\partial x}+\frac{1}{2}x^2\left(\sum_{k=1}^{\infty}u^k\right)^2\frac{\partial^2 F_2^S(x,t)}{\partial x^2}+$$

which covers the whole range of u, $0<u<1-x$. Since x is small in our region of discussion, the terms containing x^2 and higher powers of x can be neglected in the first approximation as discussed in our earlier works [1, 14-16], $F_2^S(x/w,t)$ can be approximated for small-x as

$$F_2^S(x/w,t) \cong F_2^S(x,t)+x\sum_{k=1}^{\infty}u^k\frac{\partial F_2^S(x,t)}{\partial x}. \qquad (3)$$

Similarly, $G(x/w,t)$ and $F_2^{NS}(x/w,t)$ can be approximated for small-x as

$$G(x/w,t) \cong G(x,t)+x\sum_{k=1}^{\infty}u^k\frac{\partial G(x,t)}{\partial x}.$$

and $\quad F_2^{NS}(x/w,t) \cong F_2^{NS}(x,t)+x\sum_{k=1}^{\infty}u^k\frac{\partial F_2^{NS}(x,t)}{\partial x}.$

Using eqs. (3) and (4) in eq. (1) and performing u-integration we get

$$\frac{\partial F_2^S(x,t)}{\partial t}-\left[\frac{\alpha_s(t)}{2\pi}A_1(x)+\left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_1(x)\right]F_2^S(x,t)$$

$$-\left[\frac{\alpha_s(t)}{2\pi}A_2(x)+\left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_2(x)\right]G(x,t)$$

$$-\left[\frac{\alpha_s(t)}{2\pi}A_3(x)+\left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_3(x)\right]\frac{\partial F_2^S(x,t)}{\partial x}$$

$$-\left[\frac{\alpha_s(t)}{2\pi}A_4(x)+\left(\frac{\alpha_s(t)}{2\pi}\right)^2 B_4(x)\right]\frac{\partial G(x,t)}{\partial x}=0, \quad (6)$$

where

$$A_1(x)=\frac{2}{3}\{3+4\ln(1-x)+(x-1)(x+3)\},$$

$$B_1(x)=x\int_0^1 f(w)dw-\int_0^x f(w)dw+\frac{4}{3}N_f\int_x^1 F_{qq}(w)dw,$$

$$A_2(x)=N_f\left[\frac{1}{3}(1-x)(2-x+2x^2)\right],$$

$$B_2(x)=\int_x^1 F_{qg}^S(w)\,dw,$$

$$A_3(x)=\frac{2}{3}\left\{x(1-x^2)+2x\ln\left(\frac{1}{x}\right)\right\},$$

$$B_3(x)=x\int_x^1\left\{f(w)+\frac{4}{3}N_f F_{qq}(w)\right\}\frac{1-w}{w}dw,$$

$$A_4(x)=N_f x\left\{\ln\frac{1}{x}-\frac{1}{3}(1-x)\left(5-4x+2x^2\right)\right\}$$

and

$$B_4(x)=x\int_x^1\frac{1-w}{w}F_{qg}^S(w)\,dw.$$

Let us assume for simplicity [14-16]

$$G(x,t)=K(x)\,F_2^S(x,t), \quad (7)$$

where $K(x)$ is a function of x. In this connection, earlier we considered [11] $K(x)=k, ax^b, ce^{-dx}$, where k, a, b, c, d are constants. Agreement of the results with experimental data is found to be excellent for $k=4.5$, $a=4.5$, $b=0.01$, $c=5$, $d=1$ for low-x in leading order. But correct form of $K(x)$ can actually be obtained only by solving coupled DGLAP evolution equations for singlet and gluon structure functions, and works are going on in this regard. Therefore, eq. (6) becomes

$$\frac{\partial F_2^S(x,t)}{\partial t}-\left[\frac{\alpha_s(t)}{2\pi}L_1(x)+\left(\frac{\alpha_s(t)}{2\pi}\right)^2 M_1(x)\right]F_2^S(x,t)$$

$$-\left[\frac{\alpha_s(t)}{2\pi}L_2(x)+\left(\frac{\alpha_s(t)}{2\pi}\right)^2 M_2(x)\right]\frac{\partial F_2^S(x,t)}{\partial x}=0, \quad (8)$$

where

$$L_1(x)=A_1(x)+K(x)A_2(x)+A_4(x)\frac{\partial K(x)}{\partial x},$$

$$M_1(x)=B_1(x)+K(x)B_2(x)+B_4(x)\frac{\partial K(x)}{\partial x},$$

$$L_2(x)=A_3(x)+K(x)\,A_4(x),$$

$$M_2(x)=B_3(x)+K(x)\,B_4(x).$$

For a possible solution, we assume [9, 12]

$$\left(\frac{\alpha_s(t)}{2\pi}\right)^2=T_0\left(\frac{\alpha_s(t)}{2\pi}\right), \quad (9)$$

where T_0 is a numerical parameter to be obtained from the particular Q^2-range under study. By a suitable choice of T_0 we can reduce the error to a minimum. Now, eq. (8) can be recast as

$$\frac{\partial F_2^S(x,t)}{\partial t}-P_S(x,t)\frac{\partial F_2^S(x,t)}{\partial x}-Q_S(x,t)F_2^S(x,t)=0, \quad (10)$$

where

$$P_S(x,t)=\frac{\alpha_s(t)}{2\pi}\left[L_2(x)+T_0M_2(x)\right]$$

and $$Q_S(x,t)=\frac{\alpha_s(t)}{2\pi}\left[L_1(x)+T_0M_1(x)\right].$$

Secondly, using eqs. (5) and (9) in eq. (2) and performing u-integration, we have

$$\frac{\partial F_2^{NS}(x,t)}{\partial t}-P_{NS}(x,t)\frac{\partial F_2^{NS}(x,t)}{\partial x}-Q_{NS}(x,t)F_2^{NS}(x,t)=0, \quad (11)$$

where

$$P_{NS}(x,t)=\frac{\alpha_s(t)}{2\pi}\left[A_5(x)+T_0B_5(x)\right]$$

and

$$Q_{NS}(x,t)=\frac{\alpha_s(t)}{2\pi}\left[A_6(x)+T_0B_6(x)\right]$$

with

$$A_5(x)=\frac{2}{3}\left\{x(1-x^2)+2x\ln\left(\frac{1}{x}\right)\right\},$$

$$B_5(x)=x\int_x^1\frac{1-w}{w}f(w)\,dw,$$

$$A_6(x)=\frac{2}{3}\{3+4\ln(1-x)+(x-1)(x+3)\},$$

$$B_6(x)=-\int_0^x f(w)dw+x\int_0^1 f(w)dw.$$

The general solutions [17, 18] of eq. (10) is $F(U, V) = 0$, where F is an arbitrary function and $U(x, t, F_2^S) = C_1$ and $V(x, t, F_2^S) = C_2$, where C_1 and C_2 are constant and they form a solution of equations

$$\frac{dx}{P_S(x,t)} = \frac{dt}{-1} = \frac{dF_2^S(x,t)}{-Q_S(x,t)}. \tag{12}$$

We observed that the Lagrange's auxiliary system of ordinary differential equations [17, 18] occurring in the formalism, can not be solved without the additional assumption of linearization (eq. (9)) and introduction of an ad hoc parameter T_0. This parameter does not affect the results of t-evolution of structure functions. Solving eq. (12), we obtain

$$U(x,t,F_2^S) = t^{(b/t+1)} \exp\left[\frac{b}{t} + \frac{N_S(x)}{a}\right]$$

and $V(x,t,F_2^S) = F_2^S(x,t)\exp[M_S(x)]$,

where

$$a = \frac{2}{\beta_0},\ b = \frac{\beta_1}{\beta_0^2},\ N_S(x) = \int \frac{dx}{L_2(x) + T_0 M_2(x)}$$

and $$M_S(x) = \int \frac{L_1(x) + T_0 M_1(x)}{L_2(x) + T_0 M_2(x)} dx.$$

If U and V are two independent solutions of eq. (12) and if α and β are arbitrary constants, then $V = \alpha U + \beta$ may be taken as a complete solution of eq. (12). Then the complete solution [17, 18]

$$F_2^S(x,t)\exp[M_S(x)] = \alpha\left[t^{(b/t+1)} \exp\left(\frac{b}{t} + \frac{N_S(x)}{a}\right)\right] + \beta \tag{13}$$

is a two-parameter family of planes. The one parameter family determined by taking $\beta = \alpha^2$ has equation

$$F_2^S(x,t)\exp[M_S(x)] = \alpha\left[t^{(b/t+1)} \exp\left(\frac{b}{t} + \frac{N_S(x)}{a}\right)\right] + \alpha^2. \tag{14}$$

Differentiating eq. (14) with respect to a, we obtain

$$\alpha = -\frac{1}{2} t^{(b/t+1)} \exp\left[\frac{b}{t} + \frac{N_S(x)}{a}\right].$$

Putting the value of α again in eq. (14), we obtain envelope

$$F_2^S(x,t)\exp[M_S(x)] = -\frac{1}{4}\left[t^{(b/t+1)} \exp\left(\frac{b}{t} + \frac{N_S(x)}{a}\right)\right]^2.$$

Therefore,

$$F_2^S(x,t) = -\frac{1}{4} t^{2(b/t+1)} \exp\left[\frac{2b}{t} + \frac{2N_S(x)}{a} - M_S(x)\right], \tag{15}$$

which is merely a particular solution of the general solution.

Now, defining

$$F_2^S(x,t_0) = -\frac{1}{4} t_0^{2(b/t_0+1)} \exp\left[\frac{2b}{t_0} + \frac{2N_S(x)}{a} - M_S(x)\right]$$

at $t = t_0$, where $t_0 = \ln(Q_0^2/\Lambda^2)$ at any lower value $Q = Q_0$, we get from eq. (15)

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^2 \exp\left[2b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right] \tag{16}$$

which gives the t-evolution of the singlet structure function $F_2^S(x,t)$ in NLO.

Proceeding exactly in the same way, and defining

$$F_2^{NS}(x,t_0) = -\frac{1}{4} t_0^{2(b/t_0+1)} \exp\left[\frac{2b}{t_0} + \frac{2N_{NS}(x)}{a} - M_{NS}(x)\right],$$

where $N_{NS}(x) = \int \frac{dx}{A_5(x) + T_0 B_5(x)}$

and $$N_{NS}(x) = \int \frac{A_6(x) + T_0 B_6(x)}{A_5(x) + T_0 B_5(x)} dx,$$

we get for non-singlet structure function in NLO

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^2 \exp\left[2b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right] \tag{17}$$

which gives the t-evolution of the singlet structure function $F_2^{NS}(x,t)$ in NLO for $\beta = \alpha^2$.

In an earlier communication [1], we suggested that for low-x in LO $\beta = \alpha^2$

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t}{t_0}\right)^2 \tag{18}$$

and

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t}{t_0}\right)^2 \tag{19}$$

We observe that if b tends to zero, then eqs. (16) and (17) tend to eqs. (18) and (19) respectively, *i.e.*, solution of NLO equations goes to that of LO equations. Physically, b tends to zero means that the number of flavours is high.

Again defining,

$$F_2^S(x_0,t) = -\frac{1}{4} t^{(b/t+1)} \exp\left[\frac{2b}{t} + \frac{2N_S(x)}{a} - M_S(x)\right]_{x=x_0},$$

we obtain from eq. (15)

$$F_2^S(x,t) = F_2^S(x_0,t)\exp\int_{x_0}^{x}\left[\frac{2}{a}\frac{1}{L_2(x)+T_0M_2(x)} - \frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right]dx, \tag{20}$$

which gives the x-evolution of singlet structure function $F_2^S(x,t)$ in NLO.

Similarly, defining

$$F_2^{NS}(x_0,t) = -\frac{1}{4}t^{(b/t+1)}\exp\left[\frac{2b}{t}+\frac{2N_{NS}(x)}{a}-M_{NS}(x)\right]_{x=x_0},$$

we get

$$F_2^{NS}(x,t) = F_2^{NS}(x_0,t)\exp\int\left[\frac{\;}{a\;A_5(x)+T_0B_5(x)} - \frac{A_6(x)+T_0B_6(x)}{A_5(x)+T_0B_5(x)}\right]dx, \tag{21}$$

which gives the x-evolution of non-singlet structure function $F_2^{NS}(x,t)$ in NLO.

In an earlier communication [1], we suggested that for low-x in LO for $\beta=\alpha^2$

$$F_2^S(x,t) = F_2^S(x_0,t)\exp\int_{x_0}^{x}\left(\frac{2}{A_fM(x)}-\frac{L(x)}{M(x)}\right)dx \tag{22}$$

and

$$F_2^{NS}(x,t) = F_2^{NS}(x_0,t)\exp\int\left(\frac{2}{A_fQ(x)}-\frac{P(x)}{Q(x)}\right)dx\ , \tag{23}$$

where

$$A_f = 4/(33-2N_f),\ \ P(x) = 3+4\ln(1-x)-(1-x)(x+3),$$

$$Q(x) = x\left(1-x^2\right)-2x\ln x,$$

$$L(x) = P(x)+K(x)C(x)+D(x)\frac{\partial K(x)}{\partial x}$$

and $M(x) = Q(x)+K(x)\,D(x)$,

where again,

$$C(x) = 1/2N_f(1-x)\left(2-x+2x^2\right)$$

and $D(x) = N_fx\left[-1/2(1-x)(5-4x+2x^2)+(3/2)\ln(1/x)\right]$.

Of course, unlike for the t-evolution equations, we could not have for the x-evolution equations in LO as some limiting case of NLO equations.

Deuteron, proton and neutron structure functions measured in deep inelastic electro-production, can be written in terms of singlet and non-singlet quark distribution functions [19] as

$$F_2^d(x,t) = 5/9F_2^S(x,t), \tag{24}$$

$$F_2^p(x,t) = 5/18F_2^S(x,t)+3/18F_2^{NS}(x,t), \tag{25}$$

$$F_2^n(x,t) = 5/18F_2^S(x,t)-3/18F_2^{NS}(x,t) \tag{26}$$

and $$F_2^p(x,t)-F_2^n(x,t) = 1/3F_2^{NS}(x,t). \tag{27}$$

Now using eqs. (16) and (20) in eq. (24), we will get t and x-evolution of deuteron structure function $F_2^d(x,t)$ at low-x in NLO as

$$F_2^d(x,t) = F_2^d(x,t_0)\left[\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right]\exp\ 2b\left[\frac{1}{t}-\frac{1}{t_0}\right] \tag{28}$$

and $$F_2^d(x,t) = F_2^d(x_0,t)\exp\int\left[\frac{\;}{a\;L_2(x)+T_0M_2(x)} - \frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right]dx, \tag{29}$$

where the input functions are

$$F_2^d(x,t_0) = \frac{5}{9}F_2^S(x,t_0)\ \text{ and }\ F_2^d(x_0,t) = \frac{5}{9}F_2^S(x_0,t)$$

The corresponding results for particular solutions of DGLAP evolution equations in LO for $\beta=\alpha^2$ obtained earlier [1] are

$$F_2^d(x,t) = F_2^d(x_0,t)\left(\frac{t}{\;}\right) \tag{30}$$

and

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\int_{x_0}\left(\frac{1}{A_fM(x)}-\frac{L(x)}{M(x)}\right)dx\ . \tag{31}$$

Similarly, using eqs. (16) and (17) in eqs. (25), (26) and (27), we get the t – evolutions of proton, neutron, and difference and ratio of proton and neutron structure functions at low-x in NLO as

$$F_2^p(x,t) = F_2^p(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)\exp\ 2b\left[\frac{1}{t}-\frac{1}{t_0}\right], \tag{32}$$

$$F_2^n(x,t) = F_2^n(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)\exp\left[2b\left(\frac{1}{t}-\frac{1}{t_0}\right)\right], \tag{33}$$

$$F_2^p(x,t) - F_2^n(x,t) = \left[F_2^p(x,t_0) - F_2^n(x,t_0)\right]$$

$$\left(\frac{t^{(b/t+1)}}{t^{(b/t_0+1)}}\right) \exp\left[2b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right] \tag{34}$$

and

$$\frac{F_2^p(x,t)}{F_2^n(x,t)} = \frac{F_2^p(x,t_0)}{F_2^n(x,t_0)} = R(x), \tag{35}$$

where $R(x)$ is a constant for fixed-x. The input functions are

$$F_2^p(x,t_0) = \frac{5}{18} F_2^S(x,t_0) + \frac{3}{18} F_2^{NS}(x,t_0),$$

$$F_2^n(x,t_0) = \frac{5}{18} F_2^S(x,t_0) - \frac{3}{18} F_2^{NS}(x,t_0),$$

and $\quad F_2^p(x,t_0) - F_2^n(x,t_0) = \frac{1}{3} F_2^{NS}(x,t_0).$

The corresponding results for particular solutions of DGLAP evolution equations in LO for $\beta = \alpha^2$ are

$$F_2^p(x,t) = F_2^p(x,t_0)\left|\frac{1}{t}\right|, \tag{36}$$

$$F_2^n(x,t) = F_2^n(x,t_0)\left|\frac{1}{t}\right|, \tag{37}$$

$$F_2^p(x,t) - F_2^n(x,t) = \left[F_2^p(x,t_0) - F_2^n(x,t_0)\right]\left(\frac{1}{t_0}\right)^{-}, \tag{38}$$

and $$\frac{F_2^p(x,t)}{F_2^n(x,t)} = \frac{F_2^p(x,t_0)}{F_2^n(x,t_0)} = R(x), \tag{39}$$

where $R(x)$ is a constant for fixed-x.

It is observed that the ratio of proton and neutron is same for both NLO and LO and it is independent of t for fixed-x.

For the complete solution of eq. (10), we take $\beta = \alpha^2$ in eq. (13). If we take $\beta = \alpha$ in eq. (13) and differentiate with respect to α as before, we get

$$0 = t^{(b/t+1)} \exp\left(\frac{b}{t} + \frac{N_s(x)}{a}\right) + 1,$$

from which we can not determine the value of α.

But taking $\beta = \alpha^3$ in eq. (13) and differentiating with respect to α, we get

$$\alpha = \sqrt{-\frac{1}{3} t^{(b/t+1)} \exp\left(\frac{b}{t} + \frac{N_s(x)}{a}\right)}$$

which is imaginary. Putting this value of α in eq. (13), we get ultimately

$$F_2^S(x,t) = t^{(b/t+1)3/2}\left\{\left(-\frac{1}{3}\right)^{1/2} + \left(-\frac{1}{3}\right)^{3/2}\right\}$$

$$\times \exp\left[\left(\frac{b}{t} + \frac{N_s(x)}{a}\right)^{3/2} - M_S(x)\right]$$

Now, defining

$$F_2^S(x,t_0) = t_0^{(b/t_0+1)3/2}\left\{\left(-\frac{1}{3}\right)^{1/2} + \left(-\frac{1}{3}\right)^{3/2}\right\}$$

$$\times \exp\left[\left(\frac{b}{t_0} + \frac{N_s(x)}{a}\right)^{3/2} - M_S(x)\right]$$

we get

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{3/2} \exp\left[\frac{3}{2}b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

Proceeding exactly in the same way, we get for non-singlet structure function also

$$F_2^{NS}(x,t) = F_2^{NS}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{3/2} \exp\left[\frac{3}{2}b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

Then using eqs. (24), (25), (26) and (27) we get t-evolutions of deuteron, proton, neutron and difference of proton and neutron structure functions

$$F_2^{d,p,n,p-n}(x,t) = F_2^{d,p,n,p-n}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{3/2}$$

$$\times \exp\left[\frac{3}{2}b\left(\frac{1}{t} - \frac{1}{t_0}\right)\right]$$

Proceeding in the same way, we get x-evolutions of deuteron structure function

$$F_2^d(x,t) = F_2^d(x,t_0)\exp\int_{x_0}^{x}\left(\frac{3/2}{a}\cdot\frac{1}{L_2(x) + T_0 M_2(x)}\right.$$

$$\left.-\frac{L_1(x) + T_0 M_1(x)}{L_2(x) + T_0 M_2(x)}\right) dx.$$

But the x-evolutions of proton and neutron structure functions like those of deuteron structure function can not be obtained by this methodology as discussed earlier.

Proceeding exactly in the same way, we can show that if we take $\beta = \alpha^4$, we get

$$F_2^{d,p,n,p-n}(x,t) = F_2^{d,p,n,p-n}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{4/3}$$

$$\times \exp\frac{4}{3}b\left(\frac{1}{t}-\frac{1}{t_0}\right)$$

and

$$F_2^d(x,t) = F_2^d(x,t_0)\exp\int\limits^{x}\left(\frac{4/3}{a}\cdot\frac{}{L_2(x)+T_0M_2(x)}\right.$$

$$\left.\frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right)dx.$$

Similarly, if we take $\beta = \alpha^5$, we get

$$F_2^{d,p,n,p-n}(x,t) = F_2^{d,p,n,p-n}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)^{5/4}$$

$$\times \exp\frac{5}{4}b\left(\frac{1}{t}-\frac{1}{t_0}\right)$$

and $$F_2^d(x,t) = F_2^d(x_0,t)\exp\int\limits_{x_0}^{x}\left(\frac{5/4}{a}\cdot\frac{}{L_2(x)+T_0M_2(x)}\right.$$

$$\left.-\frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right)dx.$$ and so on.

Thus we observe that if we take $\beta = \alpha$ in eq. (13), we can not obtain the value of α and also the required solution. But if we take $\beta = \alpha^2, \alpha^3, \alpha^4, \alpha^5$ ...... and so on, we see that the powers of $t^{b/t+1}/t_0^{b/t_0+1}$ and coefficient of $b(1/t - 1/t_0)$ of the exponential part in t-evolutions of deuteron, proton and neutron structure functions are 2, 3/2, 4/3, 5/4....and so on respectively, as discussed above. Similarly, for x-evolutions of deuteron structure functions we see that the numerators of the first term inside the integral sign are 2, 3/2, 4/3, 5/4....and so on respectively, for the same values of α. Thus we see that if in the relation $\beta = \alpha^y$, y varies between 2 and a maximum value, the powers of $t^{b/t+1}/t_0^{b/t_0+1}$ and coefficient of $b(1/t - 1/t_0)$ of the exponential part in t- evolution varies between 2 and 1, and the numerator of the first term in the integral sign in x-evolution varies between 2 and 1. Then it is understood that the solutions of eqs. (10) and (11) obtained by this methodology are not unique and so the t-evolutions of deuteron, proton, neutron and difference of proton and neutron structure functions, and x-evolution of deuteron structure function obtained by this methodology are also not unique. They become eqs. (28), (29), (32), (33), (34) for $y = 2$, but they reduce to equations

$$F_2^{d,p,n,p-n}(x,t) = F_2^{d,p,n,p-n}(x,t_0)\left(\frac{t^{(b/t+1)}}{t_0^{(b/t_0+1)}}\right)\exp b\left(\frac{1}{t}-\frac{1}{t_0}\right)$$

and

$$F_2^d(x,t) = F_2^d(x_0,t)\exp\int\limits_{x_0}^{x}\left(\frac{1}{a}\cdot\frac{1}{L_2(x)+T_0M_2(x)}\right.$$

$$\left.-\frac{L_1(x)+T_0M_1(x)}{L_2(x)+T_0M_2(x)}\right)dx$$

for a maximum value of y.

Thus by this methodology, instead of having a single solution, we arrive at a band of solutions, of course the range for these solutions is reasonably narrow.

3. Results and discussion

In the present paper, we compare our results of t-evolution of deuteron, proton, neutron and difference and ratio of proton and neutron structure functions with the HERA [6] and NMC [7] low-x and low-Q^2 data. In case of HERA data [6], proton and neutron structure functions are measured in the range of $2 \le Q^2 \le 50$ GeV^2. Moreover, here $P_T \le 200$ MeV, where P_T is the transverse momentum of the final state baryon. In case of NMC data, proton and deuteron structure functions are measured in the range of $0.75 \le Q_2 \le 27$ GeV^2. We consider number of flavours $N_f = 4$. We also compare our results of t-evolution of proton structure functions with recent global parameterization [8]. This parameterization includes data from H1-96\99, ZEUS-96/97(X0.98), NMC, E665 data.

In Figures 1(a-d), we present our results of t-evolutions of deuteron, proton, neutron and difference of proton and neutron structure functions (solid lines) respectively, for the representative values of x given in the figures for $y = 2$ (upper solid lines) and y = maximum (lower solid lines) in $\beta = \alpha^y$ relation. Data points at lowest-Q^2 values in the figures are taken as input to test the evolution equation. Agreement with the data [7, 6] is found to be good. In the same figures, we also plot the results of t-evolutions of deuteron, proton, neutron and difference of proton and neutron structure functions (dashed lines) for the particular solutions in leading order. Here, the upper dashed lines are for $y = 2$ and lower dashed lines, for y = maximum in $\beta = \alpha^y$ relation. We observe that t-evolutions are slightly steeper in LO calculations than those of NLO. But differences in results for proton and neutron structure functions are smaller and NLO results for $y = 2$ are in better agreement with experimental data, in general.

In Figure 2, we compare our results of t-evolutions of proton structure functions F_2^p (solid lines) with recent global parameterization [8] (long dashed lines) for the representative values of x given in the figures for $y = 2$ (upper solid lines) and y = maximum (lower solid lines) in $\beta = \alpha^y$ relation. Data points at lowest-Q^2 values in the figures are taken as input to test the evolution equation. In the same figure, we also plot the results of t-evolutions of proton structure functions F_2^p (dashed lines) for the particular solutions in leading order. Here, the upper dashed lines are for $y = 2$ and the lower dashed lines are for y = maximum in $\beta = \alpha^y$ relation. We observe that the t-evolutions are slightly steeper in LO calculations than those of NLO. Agreement with the NLO results is found to be better than with the LO results.

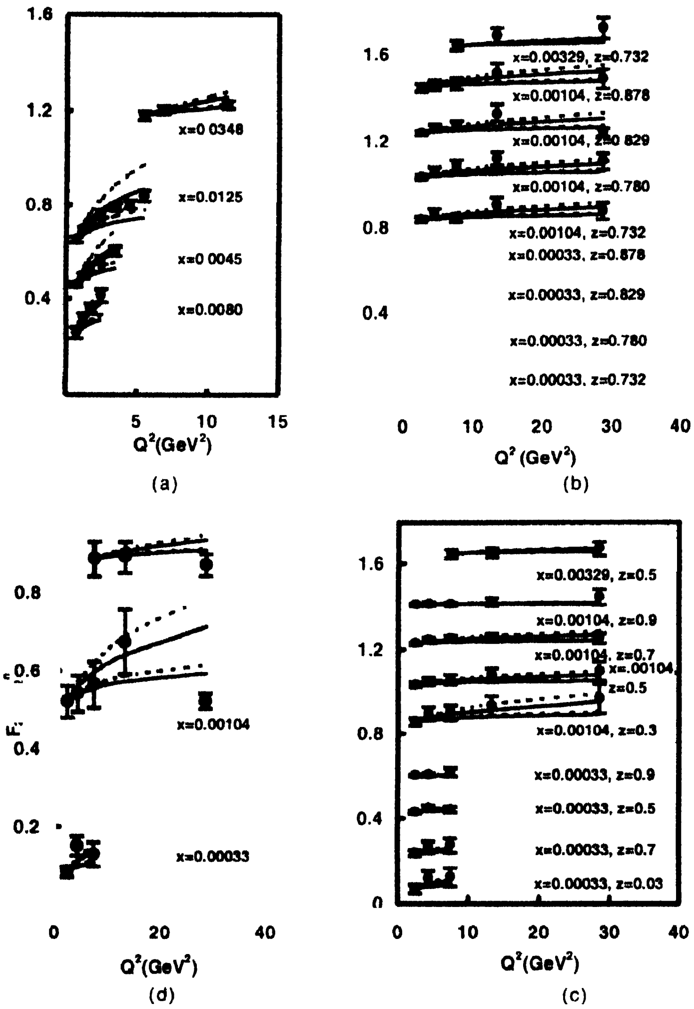

Figure 1(a-d). Results of t-evolutions of deuteron, proton, neutron and difference of proton and neutron structure functions (solid lines) for the representative values of x in next to leading order for NMC and HERA data. For convenience, value of each data point is increased by adding 0 $2i$ (a-c) and $0.4i$ (d) where i = 0, 1, 2, 3, ... are the numberings of curves counting from the bottom of the lowermost curve as the 0-th order. In the same figures, we also plot the résults of t-evolutions of deuteron, proton, neutron and difference of proton and neutron structure functions (dashed lines) for the particular solutions in leading order. Data points at lowest-Q^2 values in the figures are taken as input.

In Figure 3, we present our results of t-evolutions of ratio of proton and neutron structure functions F_2^p/F_2^n (solid lines) for the representative values of x given in the figures. Though according to our theory, the ratio should be independent of t

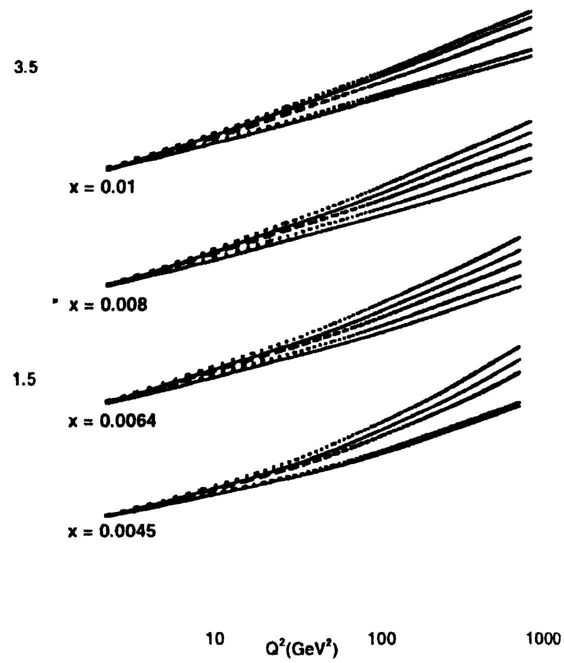

Figure 2. Results of t-evolutions of proton structure functions F_2^p (solid lines) with recent global parameterization (long dashed lines) for the representative values of x given in the figures Data points at lowest Q^2 values in the figures are taken as input In the same figure, we also plot the results of t-evolutions of proton structure functions F_2^p (dashed lines) for the particular solutions in leading order. For convenience, value of each data point is increased by adding 0 $5i$, where i = 0, 1, 2, 3, ... are the numberings of curves counting from the bottom of the lowermost curve as the 0-th order

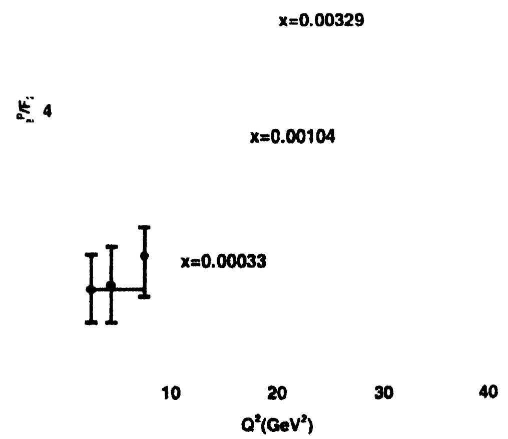

Figure 3. Results of t-evolutions of the ratio of proton and neutron structure functions F_2^p/F_2^n (solid lines) for the representative values of x given in the figures. Data points at lowest-Q^2 values in the figures are taken as input.

due to the lack of sufficient amount of data and due to large error bars, a clear cut conclusion can not be drawn.

In Figure 4, we plot $T(t)^2$ and $T_0T(t)$, where $T(t) = \alpha_s(t)/2\pi$ against Q^2 in the Q^2 range of $0.5 \le Q^2 \le 1000$ GeV^2 as required by the data used by us. Though the explicit value of T0 is not necessary in calculating t-evolution, yet we observe that for $T_0 = 0.027$, errors become minimum in the Q^2 range of $0.5 \le Q^2 \le 1000$ GeV^2.

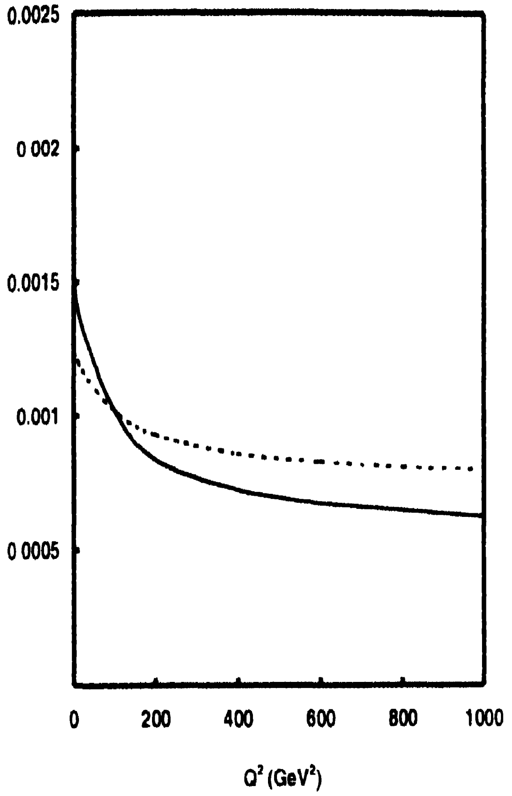

Figure 4. $T(t)^2$ and $T_0T(t)$, where $T(t) = \alpha_s(t)/2\pi$ against Q^2 in the Q^2 range of $0.5 < Q^2 \le 1000$ GeV^2

Though we compare our results for $y = 2$ and y = maximum in $\beta = \alpha^y$ relation with data, agreement of the result with experimental data is found to be excellent with $y = 2$ for t-evolution in next to leading order.

We can also calculate x-evolution of non-singlet and singlet structure function at low-x from eqs. (22) and (23). But it involves complicated integrals as eqs. (22) and (23) involve $L_1(x)$, $L_2(x)$, $M_1(x)$, and $M_2(x)$ which are again functions of $A_1(x)$, $A_2(x)$, $A_3(x)$, $A_4(x)$, $B_1(x)$, $B_2(x)$, $B_3(x)$, and $B_4(x)$. But these functions involve many integrals making the calculation of x-distribution complicated. We keep it as our subsequent work.

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